

The Basics of Digital Filters

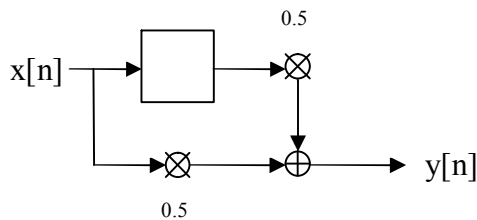
Any processing of a signal can be called filtering. The characteristics of a filter are described by its frequency response, both amplitude and phase, to sinusoids of different frequencies.

A filter can be described by an equation giving the output in terms of the input, with possible delays. This is an FIR (Finite Impulse Response) filter. If the output is dependent on the past output, then it is an IIR (Infinite Impulse Response) filter.

The simplest of all filters can be represented by the equation:

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

$x[n]$ is the n th sample in the input signal; $x[n-1]$ is the sample before it, and $y[n]$ is the output sample. Clearly the output only starts at $n = 1$, since $x[0]$ is the first sample. This can also be represented in a diagram:



The cross in the circle represents an amplifier – an amplitude changer – by the factor beside it. The plus in the circle is a summer. The delay of one sample is represented by the delta.

This simple filter can actually be analyzed exactly for its frequency response. If the input signal is sinusoidal, it can be represented by a complex exponential $e^{i\omega n}$, where n is the sample number and ω is the frequency. (We have set the sampling rate to 1, but this doesn't alter the analysis). Thus the output at frequency ω is:

$$\begin{aligned} y[n] &= 0.5(e^{i\omega n} + e^{i\omega(n-1)}) \\ &= 0.5(e^{i\omega n} + e^{i\omega n} e^{-i\omega}) \\ &= 0.5e^{i\omega n} (1 + e^{-i\omega}) \\ &= 0.5(1 + e^{-i\omega})x[n] \end{aligned}$$

The frequency response is the function of ω , i.e.

$$H(e^{i\omega}) = 0.5(1 + e^{-i\omega})$$

In polar form this is

$$H(e^{i\omega}) = G(\omega)e^{i\Theta(\omega)}$$

Where

$$G(\omega) = |H(e^{i\omega})|, \text{ and}$$

$$\Theta(\omega) = \angle H(e^{i\omega})$$

For the simple filter,

$$\begin{aligned} H(e^{i\omega}) &= \frac{1 + e^{i\omega}}{2} \\ &= e^{-i\omega/2} \left(\frac{e^{i\omega/2} + e^{-i\omega/2}}{2} \right) \\ &= e^{-i\omega/2} \cos(\omega/2) \end{aligned}$$

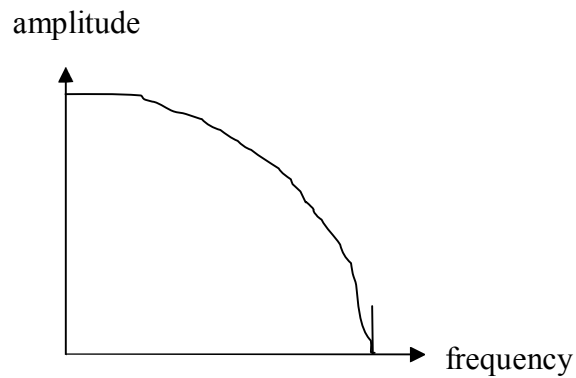
Then

$$\begin{aligned} G(\omega) &= |e^{-i\omega/2} \cos(\omega/2)| \\ &= \cos(\omega/2) \end{aligned}$$

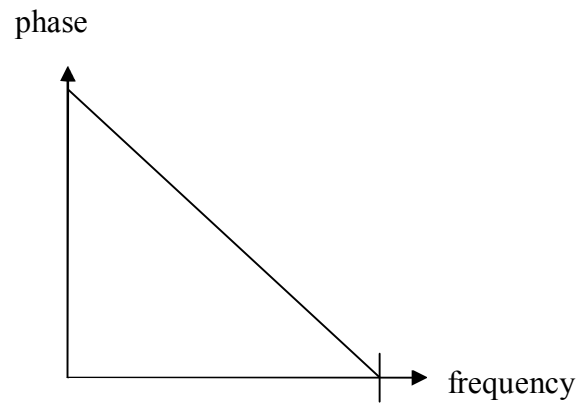
And

$$\Theta(\omega) = -\omega/2$$

The amplitude part of the frequency response thus looks like a quarter of a cos wave:



And the phase part is linear:



This is really only a lowpass filter in name only – it has no real cutoff frequency as a real lowpass filter has.

In a similar fashion, the simplest highpass filter is:

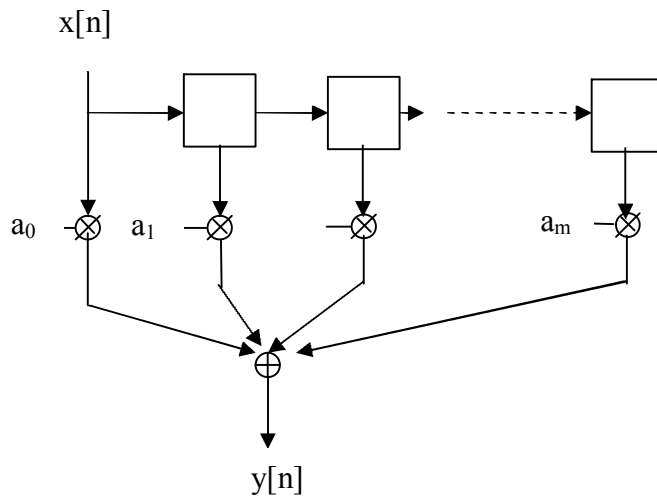
$$y[n]=0.5(x[n] - x[n-1])$$

Its amplitude response is a quarter of a sine wave, with the same linear phase.

These filters are both Finite Impulse Filters. The general form of such a filter is:

$$y[n]=a_0x[n] + a_1x[n-1] \dots a_mx[n-m]$$

which has the diagram:



Impulse Response

The response of a filter to a single pulse is a way to characterize the filter and how it works. The single impulse signal is:

$$unit[n] = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$$

A general signal, $x[n]$, can be written as:

$$x[n] = \sum_{m=0}^{\infty} x[m] \cdot unit[n - m]$$

If the output of the filter is an operator T applied to the input (i.e. $y[n]=T(x[n])$), then we have:

$$T(x[n]) = y[n] = T\left(\sum_{m=0}^{\infty} x[m] \cdot unit[n - m]\right)$$

In the summation, the $x[m]$ act as constants, so we can write this as:

$$\begin{aligned}
& T\left(\sum_{m=0}^{\infty} x[m] \cdot \text{unit}[n-m]\right) \\
&= \sum_{m=0}^{\infty} x[m] \cdot T(\text{unit}[n-m]) \\
&= \sum_{m=0}^{\infty} x[m] \cdot h[n-m]
\end{aligned}$$

H is called the impulse response – the output of the filter when the input is the unit signal. This is often written as a *convolution*:

$$(x * h)[n]$$

An explanation of convolution:

$$\begin{aligned}
& (x * \text{unit})[n] \\
&= \sum_{m=0}^{\infty} x[m] \cdot \text{unit}[n-m] \\
&= x[0] \cdot \text{unit}[n] + x[1] \cdot \text{unit}[n-1] + x[2] \cdot \text{unit}[n-2] + \dots + x[m] \cdot \text{unit}[n-m] \\
&= x[n]
\end{aligned}$$

Some equivalences concerning convolution:

$$\begin{aligned}
(x[n] * c \cdot \text{unit}[n]) &= c \cdot x[n] \\
(x[n] * \text{unit}[n-t]) &= x[n-t]
\end{aligned}$$

The convolution

$$\begin{aligned}
& (h * x)[n] \\
&= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \dots + h[m]x[n-m]
\end{aligned}$$

Which has the same form as the general FIR filter – the $h[i]$ are the a_i coefficients. Thus the impulse response h is just as good as giving the a_i coefficients.

The z-transform

Another way to analyze the response of a filter is to rewrite the Fourier transform. The complex exponential form is:

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-i\omega n}$$

We simply replace the exponential by z :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

We can think of this as a function Z applied to $x[n]$. i.e. $X=Z(x)$. In general, Z will be a polynomial in z^{-1} .

A delay of m in time becomes a multiplication by z^{-m} . This can be shown by:

$$\begin{aligned}
Z(x \text{ shifted by } \Delta) &= \sum_{n=0}^{\infty} x[n - \Delta] \cdot z^{-n} \\
&= \sum_{m=-\Delta}^{\infty} x[m] \cdot z^{-(m+\Delta)} \\
&= \sum_{m=0}^{\infty} x[m] \cdot z^{-m} \cdot z^{-\Delta}, \text{ since } x[m] = 0 \text{ for } m < 0 \\
&= z^{-\Delta} \cdot \sum_{m=0}^{\infty} x[m] \cdot z^{-m} \\
&= z^{-\Delta} \cdot X(z)
\end{aligned}$$

The z transform of a convolution is the product of the individual z transforms:

$$\begin{aligned}
Z(x * y) &= \sum_{n=0}^{\infty} (x * y)[n] \cdot z^{-n} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x[m] \cdot y[n - m] \cdot z^{-n} \\
&= \sum_{m=0}^{\infty} x[m] \cdot \sum_{n=0}^{\infty} y[n - m] \cdot z^{-n} \\
&= \sum_{m=0}^{\infty} x[m] \cdot z^{-m} \cdot Y(z) \\
&= \left(\sum_{m=0}^{\infty} x[m] \cdot z^{-m} \right) \cdot Y(z) \\
&= X(z) \cdot Y(z)
\end{aligned}$$

Thus if the output of a filter is given by $y[n] = (x * h)[n]$, then, taking the z transform of both sides we have:

$$\begin{aligned}
Y(z) &= X(z) \cdot H(z), \text{ or} \\
H(z) &= \frac{Y(z)}{X(z)}
\end{aligned}$$

So the impulse response is the z transform of the output divided by the z transform of the input. For instance, take the filter:

$$y[n] = 1/3 x[n] + 1/3 x[n - 1] + 1/3 x[n - 2]$$

The impulse response is:

$$h[n] = 1/3 \text{unit}[n] + 1/3 \text{unit}[n - 1] + 1/3 \text{unit}[n - 2]$$

The Z transform of this is:

$$\begin{aligned}
H(z) &= 1/3 + 1/3 z^{-1} + 1/3 z^{-2} \\
&= \frac{1}{3} \cdot \frac{z^2 + z + 1}{z^2} \\
&= \frac{1}{3} \cdot \frac{(z - (-1/2 + \sqrt{3}/2i))(z - (-1/2 - \sqrt{3}/2i))}{z^2}
\end{aligned}$$

There are special cases in this impulse response when $z = 0$; these are called *poles*; and when $z = -1/2 \pm \sqrt{3}/2i$; these are called *zeros*. At a pole the impulse response goes to infinity. At a zero, the response is also zero. The actual response can be plotted by substituting the complex exponential back in to the transform:

$$H(e^{i\omega}) = 1/3 + 1/3e^{-i\omega} + 1/3e^{-2i\omega}$$

amplitude

